Conformal symmetry of the Lange-Neubert evolution equation

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Abstract

The Lange-Neubert evolution equation describes the scale dependence of the wave function of a meson built of an infinitely heavy quark and light antiquark at light-like separations, which is the hydrogen atom problem of QCD. It has numerous applications to the studies of B-meson decays. We show that the kernel of this equation can be written in a remarkably compact form, as a logarithm of the generator of special conformal transformation in the light-ray direction. This representation allows one to study solutions of this equation in a very simple and mathematically consistent manner. Generalizing this result, we show that all heavy-light evolution kernels that appear in the renormalization of higher-twist B-meson distribution amplitudes can be written in the same form.

1. Studies of heavy meson weak decays have been instrumental to uncover the flavor sector of the Standard model and can be a gate to new physics at TeV scales, if it exists. Considerable effort has been invested to understand the QCD dynamics of heavy meson decays in the heavy quark limit. The B-meson distribution amplitude (DA), first introduced in [1], provides the key nonperturbative input in the QCD factorization approach [2] for weak decays involving light hadrons in the final state.

Following an established convention we define the B-meson DA as the renormalized matrix element of the bilocal operator built of an effective heavy quark field $h_v(0)$ and a light antiquark $\bar{q}(zn)$ at a light-like separation:

$$\langle 0|\bar{q}(zn)\, \not{n}[zn,0]\Gamma h_v(0)|\bar{B}(v)\rangle \;=\; -\frac{i}{2}F(\mu)\,\mathrm{Tr}\left[\gamma_5\, \not{n}\Gamma P_+\right]\,\Phi_+(z,\mu) \tag{1}$$

with

$$[zn,0] \equiv \mathrm{Pexp}\left[ig\int_0^1 d\alpha\, n_\mu A^\mu(\alpha z n)\right]. \tag{2}$$

Here v_μ is the heavy quark velocity, n_μ is the light-like vector, $n^2=0$, such that $n\cdot v=1$, $P_+=\frac{1}{2}(1+\not{v})$ is the projector on upper components of the heavy quark spinor, Γ stands for an arbitrary Dirac structure, $|\bar{B}(v)\rangle$ is the $\bar{B}$-meson state in the heavy quark effective theory

(HQET) and $F(\mu)$ is the decay constant in HQET, which is used for normalization. The effective heavy quark can be related to the Wilson line through the following equation [3]:

$$\langle 0|h_v(0)|h,v\rangle = [0,-v\infty] = \mathrm{Pexp}\left[ig\int_{-\infty}^{0} d\alpha\, v_\mu A^\mu(\alpha v)\right], \tag{3}$$

so that the operator in Eq. (1) can be viewed as a single light antiquark attached to the Wilson line with a cusp containing one lightlike and one timelike segment.

The invariant function $\Phi_+(z,\mu)$ where z is a real number defines what is usually called the leading twist B-meson DA in position space. Its Fourier transform is

$$\begin{aligned} \phi_+(k,\mu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, \mathrm{e}^{ikz}\,\Phi_+(z-i0,\mu)\,, \\ \Phi_+(z,\mu) &= \int_0^\infty dk\, \mathrm{e}^{-ikz}\,\phi_+(k,\mu)\,, \end{aligned} \tag{4}$$

where in the first equation the integration contour goes below the singularities of $\Phi_+(z,\mu)$ that are located in the upper-half plane. The parameter μ is the renormalization (factorization) scale. We tacitly imply using dimensional regularization with modified minimum subtraction.

The scale dependence of the DA is driven by the renormalization of the corresponding nonlocal operator

$$O_+(z) = \bar q(zn)\, \not{n}[zn,0]\Gamma\, h_v(0).$$

The corresponding one-loop Z-factor was computed by Lange and Neubert (LN) [4], giving rize to an evolution equation which is convenient to write, for our purposes, as a renormalization group equation for the operator $O_+(z)$ [5, 6]:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s C_F}{\pi}\mathcal{H}\right) O_+(z,\mu) = 0\,, \tag{5}$$

where

$$[\mathcal{H}f](z) = \int_0^1 \frac{d\alpha}{\alpha}\Big(f(z) - \bar\alpha f(\bar\alpha z)\Big) + \ln(i\mu z)\, f(z) - \frac{5}{4}\, f(z)\,, \qquad \bar\alpha \equiv 1-\alpha\,. \tag{6}$$

This equation thus governs the scale dependence of the B-meson DA in position space, $\Phi_+(z,\mu)$. It is fully equivalent to the original LN equation for the DA in momentum space, $\phi_+(k,\mu)$, as it is easy to show by Fourier transformation.

2. We will demonstrate that the LN kernel (6) can be written in terms of the generators of collinear conformal transformations

$$S_+ = z^2\partial_z + 2jz\,, \qquad S_0 = z\partial_z + j\,, \qquad S_- = -\partial_z\,, \tag{7}$$

where $j = 1$ is the conformal spin of the light quark. They satisfy the standard $SL(2)$ commutation relations

$$[S_+, S_-] = 2S_0\,, \qquad [S_0, S_\pm] = \pm S_\pm\,. \tag{8}$$

The starting observation is that the integral operator $\mathcal{H}$ (LN kernel) can be written in a somewhat different form by studying its action on the test functions $f(z) = z^p$, $z\partial_z f(z) = pf(z)$. Here and below $\partial_z = \partial/\partial z$. In this way one obtains

$$[\mathcal{H}f](z) = \left[\psi(z\partial_z + 2) - \psi(1) + \ln(i\mu z) - \frac{5}{4}\right] f(z) \,. \tag{9}$$

Next, we use the identity for a fractional derivative $(i\partial_z)^a$ defined as the multiplication operator k^a in momentum representation [7]:

$$(i\partial_z)^a = (iz)^{-a}\frac{\Gamma(a - z\partial_z)}{\Gamma(-z\partial_z)}. \tag{10}$$

It holds for the functions $f(z)$ that are holomorphic in the lower complex half-plane $\Im m\, z < 0$, $z \in \mathbb{C}_-$, and vanish at infinity. Fourier transform for such functions goes over positive momenta $f(z) = \int_0^\infty dk\, e^{-ikz}\tilde{f}(k)$, $(i\partial_z)^a f(z) = \int_0^\infty dk\, e^{-ikz} k^a\, \tilde{f}(k)$, corresponding in our case to positive values of the light-quark energy $\omega = k/2$ in the B-meson rest frame, cf. Eq. (4). Expanding this identity around $a = 0$ one gets

$$\ln(i\partial_z) = \psi(-z\partial_z) - \ln(iz) \tag{11}$$

and making an inversion $z \to -1/z$

$$\ln(iz^2\partial_z) = \psi(z\partial_z) + \ln(iz) \,. \tag{12}$$

Finally, since for any function $f(z\partial_z)z = zf(z\partial_z + 1)$, we can write this identity as

$$z^{-2}\ln(iz^2\partial_z)z^2 = \ln\left[i(z^2\partial_z + 2z)\right] = \ln(iS^+) = \psi(z\partial_z + 2) + \ln(iz) \,. \tag{13}$$

Comparing with Eq. (6) we see that

$$\mathcal{H} = \ln(i\mu\, S^+) - \psi(1) - \frac{5}{4} \tag{14}$$

which is our main result. Note that the scale μ under the logarithm is necessary simply because S_+ has dimension $[\text{mass}]^{-1}$.

Alternatively, the same expression can be derived starting from the commutation relations for the LN kernel obtained in Ref. [6]:

$$[S_+, \mathcal{H}] = 0 \,, \qquad [S_0, \mathcal{H}] = 1 \,. \tag{15}$$

Since the problem has one degree of freedom — the light-cone coordinate of the light quark — it follows from $[S_+, \mathcal{H}] = 0$ that the operator $\mathcal{H}$ must be a *function* of S_+, $\mathcal{H} = h(S_+)$. This function can be found using the second commutation relation. Let $S = S_0 + 1$. Then $S_+ = zS$ and the relation $[S_0, h(S_+)] = 1$ can be written equivalently as $[S, h(z\,S)] = 1$. Taking into account that $[S, z\,S] = z\,S$ one obtains an equation on the function $h(s)$

$$s\,h'(s) = 1 \implies h(s) = \ln s + \text{constant}\,, \tag{16}$$

reproducing the result in Eq. (14) up to a (scheme-dependent) constant.

3. The main advantage of Eq. (14) is that diagonalization of the kernel $\mathcal{H}$ can be traded for a much simpler task of diagonalization of the first-order differential operator S_+ (7). Eigenfunctions of S_+ take a simple form[1]

$$Q_s(z) = -\frac{1}{z^2}e^{is/z}\,, \qquad\qquad iS_+\,Q_s(z) = s\,Q_s(z)\,, \tag{17}$$

so that

$$\mathcal{H}\,Q_s(z) = \Big[\ln(\mu\,s) - \psi(1) - \frac{5}{4}\Big]\,Q_s(z)\,. \tag{18}$$

A further advantage is that one can use $SL(2)$ representation theory methods to work with these solutions, see e.g. Ref. [11] for a short discussion of this technique. In particular one can make use of the standard $SL(2)$ invariant scalar product [12] (for spin $j=1$)

$$\langle\Phi|\Psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}_-} d^2z\,\overline{\Phi(z)}\,\Psi(z)\,, \tag{19}$$

where the (two-dimensional) integration goes over the lower half-plane $\mathbb{C}_-$, $\Im m\,z<0$. The generator iS^+ is self-adjoint w.r.t. this scalar product. The eigenfunctions (17) are orthogonal to each other and form a complete set

$$\langle Q_{s'}|Q_s\rangle = \frac{1}{s}\delta(s-s')\,, \qquad\qquad \int_0^\infty ds\,s\,Q_s(z)\overline{Q_s(z')} = \frac{e^{-i\pi}}{(z-\bar z')^2}\,. \tag{20}$$

The function on the r.h.s. of the completeness relation is called reproducing kernel [13]. It acts as a unit operator so that for any function holomorphic in the lower half plane

$$\Psi(z) = \frac{1}{\pi}\int_{\mathbb{C}_-} d^2z'\,\frac{e^{-i\pi}}{(z-\bar z')^2}\Psi(z')\,. \tag{21}$$

Hence the B-meson DA (1) can be expanded as

$$\Phi_+(z,\mu) = \int_0^\infty ds\,s\,\eta(s,\mu)\,Q_s(z) = -\frac{1}{z^2}\int_0^\infty ds\,s\,e^{is/z}\,\eta(s,\mu)\,, \qquad \eta(s,\mu) = \langle Q_s|\Phi\rangle\,. \tag{22}$$

The integration goes over all possible eigenvalues of the step-up generator S_+ that corresponds to special conformal transformations along the light-ray n^μ. This representation is very similar to the one suggested in Ref. [8].

The scale-dependence of the coefficients $\eta(s,\mu)$ is governed by the renormalization-group equation

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \Gamma_{cusp}(\alpha_s)\ln(\mu\,s/s_0)\Big)F(\mu)\eta(s,\mu) = 0\,, \tag{23}$$

[1]The sign is chosen such that $Q_s(z)$ are real and positive for $z=-i\tau$, $\tau>0$.

where $s_0 = e^{5/4-\gamma_E}$ and $\Gamma_{cusp}(\alpha_s) = \frac{\alpha_s}{\pi} C_F + \ldots$ is the cusp anomalous dimension [9, 10].

The solution of this equation takes the form

$$
\begin{aligned}
F(\mu)\,\eta(s,\mu) &= F(\mu_0)\,\eta(\xi,\mu_0) \times \exp\left\{-\int_{\mu_0}^{\mu} \frac{d\tau}{\tau}\,\Gamma_{cusp}(\alpha_s(\tau))\,\ln(\tau\, s/s_0)\right\} \\
&= F(\mu_0)\,\eta(\xi,\mu_0) \left(\frac{\mu_0\, s}{s_0}\right)^{r(\mu)} B(\mu)\,,
\end{aligned}
\tag{24}
$$

where

$$
\begin{aligned}
r(\mu) &= -\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)}\,\Gamma_{cusp}(\alpha_s) = 2C_F/\beta_0\,\ln\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right) + \ldots\,, \\
B(\mu) &= \exp\left\{-\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)}\,\Gamma_{cusp}(\alpha) \int_{\alpha(\mu_0)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}\right\}.
\end{aligned}
\tag{25}
$$

In practical applications the momentum (energy) representation for the B-meson DA $\phi_+(k,\mu)$ as defined in (4) is more convenient. This can be derived easily by observing that exponential functions e^{-ipz}, $p > 0$ are mutually orthogonal and form a complete set w.r.t. the same scalar product

$$
\langle e^{-ipz} | e^{-ip'z} \rangle = \frac{1}{p}\delta(p - p')\,.
\tag{26}
$$

Hence

$$
\Phi_+(z,\mu) = \int_0^\infty dp\, p\, e^{-ipz} \langle e^{-ipz} | \Phi_+(z,\mu)\rangle = \int_0^\infty dp\, p\, e^{-ipz} \int_0^\infty ds\, s\,\eta(s,\mu)\,\langle e^{-ipz} | Q_s(z)\rangle
\tag{27}
$$

and therefore

$$
\phi_+(k,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\; \mathrm{e}^{ikz}\,\Phi_+(z - i0,\mu) = k \int_0^\infty ds\, s\,\eta(s,\mu)\,\langle e^{-ikz} | Q_s(z)\rangle\,.
\tag{28}
$$

Using

$$
\langle e^{-ikz} | Q_s(z)\rangle = \frac{1}{\sqrt{ks}} J_1(2\sqrt{ks})
\tag{29}
$$

we finally obtain

$$
\phi_+(k,\mu) = \int_0^\infty ds\,\sqrt{ks}\, J_1(2\sqrt{ks})\,\eta(s,\mu)\,,
\tag{30}
$$

where $J_1(x)$ is the Bessel function. The representation in Eq. (30) is equivalent to the one suggested by Bell, Feldmann, Wang and Yip in Ref. [8], who noticed that the evolution equation is significantly simplified in this manner. In their notation, cf. second line in Eq. (2.17), $s\,\eta(s,\mu) \equiv \rho_+(1/s,\mu)$.

The orthogonality relation (26) combined with the projection (29) leads to a familiar relation for the Bessel functions

$$\int_0^\infty ds\, J_1(2\sqrt{ps})\, J_1(2\sqrt{p's}) = \delta(p-p')\,, \tag{31}$$

which can be used to invert Eq. (30) and express $\eta(s,\mu)$ in terms of $\phi_+(k,\mu)$.

Note that the representation in (14) is valid for the evolution kernel in momentum space as well, but the generator S_+ has to be taken in the adjoint representation

$$\mathcal{S}_+ = i\big[k\partial_k^2 + 2j\partial_k\big]\,, \qquad j=1\,. \tag{32}$$

The Bessel functions appearing in (29), (30) are eigenfunctions of $\mathcal{S}_+$, indeed:

$$s\langle e^{-ikz}|Q_s(z)\rangle = \langle e^{-ikz}|iS_+Q_s(z)\rangle = \langle iS_+e^{-ikz}|Q_s(z)\rangle = i\mathcal{S}_+\langle e^{-ikz}|Q_s(z)\rangle\,. \tag{33}$$

Of particular interest for the QCD description of B-decays is the value of the first negative moment

$$\lambda_B^{-1}(\mu) = \int_0^\infty \frac{dk}{k}\phi_+(k,\mu) = \int_0^\infty d\tau\, \Phi_+(-i\tau,\mu) = \int_0^\infty ds\, \eta(s,\mu)\,. \tag{34}$$

As demonstrated in [8], QCD factorization expressions for B decay amplitudes can conveniently be written in terms of $\eta(s,\mu)$ as well, so that we do not dwell on this topic here.

5. The same representation can be derived for arbitrary two-particle heavy-light one-loop kernels that contribute to the evolution equations for higher-twist B-meson DAs [6]. The difference to the leading twist is that the two-particle evolution equations are not closed: The two-particle, $2\to 2$, kernels appear as parts of larger mixing matrices involving $2\to 3$ parton transitions, however, $3\to 2$ transitions do not occur at the one-loop level.

Explicit expressions for all $2\to 2$ heavy-light kernels have been derived in Ref. [6], see Sec. 3.2. They can be written in terms of an integral operator

$$[\mathcal{H}_j f](z) = \int_0^1 \frac{d\alpha}{\alpha}\Big[f(z) - \bar\alpha^{2j-1}f(\bar\alpha z)\Big] + \ln(i\mu z)f(z) - [\sigma_h+\sigma_\ell]f(z)\,, \tag{35}$$

where j is the conformal spin of the light parton ℓ (quark or gluon) and the constants $\sigma_h = 1/2$, $\sigma_{\rm quark} = 3/4$, $\sigma_{\rm gluon} = \beta_0/4N_c$ ($\beta_0 = 11/3N_c - 2/3n_f$) are related to the anomalous dimensions of the fields. Conformal spin of a parton is defined as $j=(d+s)/2$ where d is canonical dimension and s is spin projection on the light cone, see [14]. For a quark $j=1$ for the "plus" projection that contributes to the leading-twist B-meson DA (1), in which case (35) reproduces (6), and $j=1/2$ for the "minus" projection that is relevant for the DA $\Phi_-(z,\mu)$, cf. [2]. In turn, for a gluon $j=3/2$ for the leading-twist projection and $j=1$ for the higher-twist.

Following the above derivation for $j=1$ we obtain the following representation for the kernel in the general case:

$$\mathcal{H}_j = \ln(i\mu S_+^{(j)}) - \psi(1) - \sigma_h - \sigma_\ell\,, \tag{36}$$

where the generator of special conformal transformations $S_+^{(j)}$ for spin j is defined in Eq. (7). The eigenfunctions of $S_+^{(j)}$ have the form

$$Q_s^{(j)}(z) = \frac{e^{-i\pi j}}{z^{2j}} e^{is/z}\,, \qquad\qquad iS_+^{(j)}\, Q_s^{(j)}(z) = s\, Q_s^{(j)}(z)\,. \tag{37}$$

They are orthogonal and form a complete set with respect to the $SL(2)$ scalar product [13]

$$\langle\Phi|\Psi\rangle_j = \frac{2j-1}{\pi}\int_{\mathbb{C}_-} \mathcal{D}_j z\, \overline{\Phi(z)}\, \Psi(z)\,, \tag{38}$$

where $\mathcal{D}_j z = d^2 z\, [i(z-\bar{z})]^{2j-2}$. One obtains

$$\langle Q_s^{(j)}|Q_{s'}^{(j)}\rangle_j = \frac{\Gamma(2j)}{s^{2j-1}}\,\delta(s-s')\,, \qquad \frac{1}{\Gamma(2j)}\int_0^\infty ds\, s^{2j-1}\, Q_s^{(j)}(z)\, \overline{Q_s^{(j)}(z')} = \frac{e^{-i\pi j}}{(z-\bar{z}')^{2j}}\,. \tag{39}$$

The expression on the r.h.s. of the second integral defines the reproducing kernel for arbitrary spin j [13], i.e. for arbitrary function (holomorphic in the lower plane)

$$\Psi(z) = \frac{2j-1}{\pi}\int_{\mathbb{C}_-} \mathcal{D}_j z\, \frac{e^{-i\pi j}}{(z-\bar{z}')^{2j}}\, \Psi(z)\,. \tag{40}$$

The functions $Q_s^j(z)$ diagonalize the renormalization group kernel

$$\mathcal{H}_j Q_s^j(z) = \big[\ln(\mu s) - \psi(1) - \sigma_h - \sigma_\ell\big] Q_s^j(z) \tag{41}$$

so that it is natural to write matrix elements of generic heavy-light operators as an expansion

$$\Phi_j(z,\mu) = \int_0^\infty ds\, s^{2j-1} \eta_j(s,\mu)\, Q_s^{(j)}(z)\,, \tag{42}$$

where $\Phi_j(z,\mu)$ is analogue of $\Phi_+(z,\mu)$ (1).

The expansion coefficients $\phi_j(k,\mu)$ appearing in the Fourier transform

$$\Phi_j(z,\mu) = \int_0^\infty dk\, e^{-ikz}\, \phi_j(k,\mu) \tag{43}$$

can be found making use of the following relations:

$$\begin{aligned} \langle e^{-ikz}|e^{-ik'z}\rangle_j &= \Gamma(2j)\, k^{1-2j}\, \delta(k-k')\,, \\ \langle e^{-ikz}|Q_s^{(j)}\rangle_j &= \Gamma(2j)\, (ks)^{1/2-j}\, J_{2j-1}(2\sqrt{ks})\,. \end{aligned} \tag{44}$$

In this way one obtains

$$\phi_j(p,\mu) = \int_0^\infty ds\, \eta_j(s,\mu)\, (sp)^{j-1/2}\, J_{2j-1}(2\sqrt{ps})\,. \tag{45}$$

In particular for $j = 1/2$ corresponding to the B-meson DA $\phi_-(k,\mu)$ [2] the conformal expansion goes over Bessel functions $J_0(2\sqrt{ks})$ as compared to $J_1(2\sqrt{ks})$ for the leading twist, cf. [8].

6. To summarize, we have constructed a conformal expansion of the distribution amplitudes of heavy-light mesons in terms of eigenfunctions of the generator of special conformal transformations. This construction is similar in spirit to the well-known expansion of DAs of light mesons in Gegenbauer polynomials which are eigenfunctions of two-particle $SL(2)$ Casimir operators, see e.g. [5]. Similar to the latter case, this expansion can serve as a basis for the construction of approximations of phenomenological relevance.

As we have shown, this expansion is a consequence of the commutation relations (15) and it would be very interesting to find out whether these relations hold true to all orders in perturbation theory for a conformal theory like $N = 4$ SYM. The consequences of our results for the DAs of baryons made of one heavy and two light quarks should be studied as well.

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